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GENERALIZATION OF THE MODIFIED BESSEL FUNCTION AND ITS GENERATING FUNCTION

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Abstract

This paper presents new generalizations of the modified Bessel function and its generating function. This function has important application in the transient solution of a queueing system.

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1. Introduction

Generating functions play an important role in the investigation of various problems (including, for example, queueing theory and related stochastic processes), see [13].

In particular, a transient solution to an M/M/1 queueing system is based on the following well-known generating function (see [11]):

$$F_1(t) = \exp\left\{(t+\frac{1}{t})\frac{z}{2}\right\} = \sum_{n=-\infty}^{\infty} t^n I_n(z), \tag{1.1}$$

where

$$I_n(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{n+2m}}{m!\Gamma(n+m+1)},\ z \in C \tag{1.2}$$

is the modified Bessel function.

It can be shown (see [4],[12]) that in the theory of the more general M/Ek/1 queue, which reduces to the M/M/1 model for k=1, an important role is played by the extended generating function

$$F_k(t) = \exp\left\{(t + \frac{1}{t^k})\frac{z}{2}\right\} = \sum_{n=-\infty}^{\infty} t^n f_n(z). \tag{1.3}$$

The analytical form of the coefficient set $\{f_n(z)\}_{n=-\infty}^{\infty}$ has potential interest in the study of the transient solution to the M/Ek/1 system. In particular, by deriving the generating function (1.3) we can obtain the time-dependent solution for the Erlang queueing model, see [3]. It is important to present the generating function in the form of a double sum with corresponding power.

In this note we introduce a generalization of the modified Bessel function (1.2) and study some similarities to the properties of the modified Bessel function. Other generalizations of the modified Bessel function, which is related to the well-known Wright function, were introduced by Luchak, see [8],[9]. Luchak also studies batch queues in terms of this function.

2. Generalized modified Bessel functions

In this section we review some known results about the modified Bessel function and its generalization introduced by Luchak [8],[9] (see also [14]). We also introduce a new generalization of the modified Bessel function.

Note that the gamma function $\Gamma(z)$ is a meromorphic function of z (that is analytical everywhere in the bounded complex plane, except at poles) and its reciprocal

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}, \tag{2.1}$$

(where $\gamma = 0.577721566\ldots$ is Euler's constant) is an entire function. If $\mathrm{Re} z > 0$, then

$$\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx \tag{2.2}$$

and $\Gamma(n+1) = n!$ for integers $n = 1, 2, 3\ldots$

In the following text we use the factorial notation $\alpha! = \Gamma(\alpha+1)$, where α is not a positive integer. The function $\Gamma(z)$ has only single poles at the points $z = -n$, $n = 0, 1, 2, \ldots$

The modified Bessel function of the first kind and order n is defined by (1.2) and is a solution of the modified Bessel equation

$$z^2\frac{d^2}{dz^2}w+z\frac{d}{dz}w-(z^2+n^2)w=0. \tag{2.3}$$

It is known that the generating function takes the form (see [15]), given in (1.1).

We list some properties of the modified Bessel function:

$$\begin{array}{ll}
1) & I_{n-1}(z)+I_{n+1}(z)=2I_n'(z);\\
2) & I_{n-1}(z)-I_{n+1}(z)=\frac{2n}{z}I_n(z);\\
3) & zI_n'(z)+nI_n(z)=zI_{n-1}(z);\\
4) & zI_n'(z)-nI_n(z)=zI_{n+1}(z);\\
5) & \left(\frac{d}{zdz}\right)^m\{z^nI_n(z)\}=z^{n-m}I_{n-m}(z);\\
6) & \left(\frac{d}{zdz}\right)^m\left\{\frac{I_n(z)}{z^n}\right\}=\frac{I_{n+m}(z)}{z^{n+m}};\\
7) & I_0'(z)=I_1(z);\\
8) & I_{-n}(z)=I_n(z).
\end{array} \tag{2.4}$$

The following generalization of the modified Bessel function (2.1) was proposed by Luchak (see [8],[9]):

$$I_n^k(z)=\left(\frac{z}{2}\right)^n\sum_{r=0}^{\infty}\frac{(\frac{z}{2})^{r(k+1)}}{r!\Gamma(n+rk+1)}, \tag{2.5}$$

$$z\in C,\ n=0,1,2\ldots,\ k\in\{1,2\ldots\}.$$

In our terminology the function (2.5) is called the *generalized modified Bessel function of the first type*. The function $I_n^k(z)$ reduces to the Bessel function $I_n(z)$ when $k=1$. Luchak derived the following properties of this function:

1)

$$I_n^k(z)=\frac{\left(\frac{1}{2}z\right)^n}{2\pi i}\int^{(0+)}x^{-n-1}\exp\left(x+\frac{\left(\frac{1}{2}z\right)^{k+1}}{x^k}\right)dx;$$

where the contour of integration is any loop around the origin once clock wise;

2)

$$I_{n+k}^k(z)=\frac{1}{k}\left(I_{n-1}^k(z)-\frac{2n}{z}I_n^k(z)\right);$$

3)
$$\frac{d}{dz}\left(z^{-n}I_n^k(z)\right) = \left(\frac{1}{2}(k+1)\right) z^{-n} I_{n+k}^k(z);$$

4)
$$\frac{d}{dz}I_n^k(z) = \frac{1}{k}\left[\left(\frac{1}{2}(k+1)\right) I_{n-1}^k(z) - \frac{n}{z}I_n^k(z)\right];$$

5)
$$\frac{d}{dz}\left(z^n I_n^k(z)\right) = z^n \left[\left(\frac{k-1}{k}\right)\left(\frac{n}{z}\right) I_n^k(z) + \left(\frac{k+1}{2k}\right) I_{n-1}^k(z)\right];$$

6) the series defining $I_n^k(z)$ converges for all z.

The differential equation satisfied by $I_n^k(z)$ is

$$\left(\frac{1}{2}(k+1)\right) z^{k-n} I_{n+k}^k(z)$$
$$= \frac{d}{dz}\left\{\left(\frac{2k}{k+1}\right)^k z^{2k-2n} \prod_{s=0}^{k-1}\left[\frac{1}{t}\frac{d}{dz} - \frac{k+1}{k}\frac{n-s}{z^2}\right] z^n I_n^k(z)\right\}.$$

Note that, both the modified Bessel function (1.2) and the generalized Bessel function of the first type (2.5) can be expressed in terms of the Wright function:

$$W(z,\rho,\beta) = \sum_{r=0}^{\infty}\frac{z^r}{r!\Gamma(\rho r+\beta)},\ z\in C,\ \beta\in C,\ \rho>-1, \tag{2.6}$$

which was introduced by E.W. Wright (see [16]).

Indeed, in terms of the Wright function, $I_n(z)$ and $I_n^k(z)$ have the form

$$I_n(z) = \left(\frac{z}{2}\right)^n W\left(\frac{z^2}{4}, 1, n+1\right), \tag{2.7}$$

$$I_n^k(z) = \left(\frac{z}{2}\right)^n W\left(\frac{z^{k+1}}{2^{k+1}}, k, n+1\right). \tag{2.8}$$

Further properties of the Wright function can be found in [2].

We now introduce a new generalization of the modified Bessel function (1.2). Consider the function

$$\tilde{I}_n^{k,s}(z) = \left(\frac{z}{2}\right)^{n+k-s}\sum_{r=0}^{\infty}\frac{(\frac{z}{2})^{r(k+1)}}{(k(r+1)-s)!\Gamma(n+r+1)}, \tag{2.9}$$

where $z \in C,\ s \in \{1,2,\ldots,k\}\,,\ n = 0, \pm 1, \pm 2, \ldots,\ k = \{1,2,\ldots\}$.

Note that the series (2.9) converges absolutely for all $z \in C$ (see below, Section 3). The function $\tilde{I}_n^{k,s}(z)$ reduces to the modified Bessel function (1.2), when $k = s = 1$.

In our terminology the function (2.9) is called the *generalized modified Bessel function of the second type*.

We are now able to generalize the formula (1.1).

THEOREM 1. *The generating function of the generalized modified Bessel function of the second type takes the form*

$$e^{\frac{1}{2}z(y^k+\frac{1}{y})} = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} y^{k(n-1)+s} \tilde{I}_n^{k,s}(z). \tag{2.10}$$

P r o o f. By using the Laurent expansion, we obtain

$$e^{\frac{1}{2}z(y^k+\frac{1}{y})} = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}zy^k)^m}{m!} \frac{(\frac{1}{2}\frac{z}{y})^r}{r!}. \tag{2.11}$$

We need the following identity (see, for example, [13]):

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi(m,n) = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi(mM+j, nN+k),\ M, N \in \{1,2,\ldots\}\,, \tag{2.12}$$

where ψ is arbitrary function such that the series (2.12) converge. Putting $M = 1,\ n = r,\ k = l$ we get

$$\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \psi(m,r) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^{k} \psi(m, rk+l-1). \tag{2.13}$$

Thus, (2.11) can be rewritten as

$$\begin{aligned}\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}zy^k)^m}{m!} \frac{(\frac{1}{2}\frac{z}{y})^r}{r!} &= \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^{k} \frac{(\frac{1}{2}zy^k)^m}{m!} \frac{(\frac{1}{2}\frac{z}{y})^{rk+l-1}}{(rk+l-1)!} \\ &= \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^{k} \frac{(\frac{1}{2}z)^{m+rk+l-1} y^{km-rk-l+1}}{m!(rk+l-1)!}. \end{aligned} \tag{2.14}$$

Changing the variables $n = m - r,\ n \in (-\infty, \infty)$ in (2.14) gives

$$\sum_{n=-\infty}^{\infty} \sum_{r=0}^{\infty} \sum_{l=1}^{k} \frac{(\frac{1}{2}z)^{n+r+rk+l-1} y^{kn-l+1}}{(r+n)!(rk+l-1)!}.$$

If we change the variables $s = k - l + 1,\ s = 1, \ldots, k$, we obtain

$$\begin{aligned} e^{\frac{1}{2}z(y^k+\frac{1}{y})} &= \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} y^{k(n-1)+s} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{n+r(k+1)+k-s}}{(r+n)!(k(r+1)-s)!} \\ &= \sum_{n=-\infty}^{\infty} \sum_{s=1}^{k} y^{k(n-1)+s} \tilde{I}_n^{k,s}(z). \end{aligned} \tag{2.15}$$

■

We emphasize that it is exactly this form of the generating function with appropriate power we need for the solution of the Erlang queue, see [3].

Note that (2.9) can not be expressed in terms of the Wright function. However we are able to find a property analogous to 8) in (2.4).

Let us consider (2.9) for positive integers $n = 1, 2, \ldots$. Then for $p = -n$ we obtain

$$\tilde{I}_{-n}^{k,s}(z) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{-n+r(k+1)+k-s}}{\Gamma(r-n+1)(k(r+1)-s)!} = \sum_{r=n}^{\infty} \frac{(\frac{1}{2}z)^{-n+r(k+1)+k-s}}{\Gamma(r-n+1)(k(r+1)-s)!}.$$

Since $\frac{1}{\Gamma(r-n+1)(k(r+1)-s)!} = 0$ (by the definition (2.1)), for r such that $r - n + 1 \leq 0$, we obtain for $l = r - n$:

$$\begin{aligned} \tilde{I}_{-p}^{k,s}(z) &= \sum_{l=0}^{\infty} \frac{(\frac{1}{2}z)^{-p+(l+p)(k+1)+k-s}}{\Gamma(l+1)(k(l+p+1)-s)!} \\ &= \left(\frac{z}{2}\right)^{k-s+pk} \sum_{l=0}^{\infty} \frac{(\frac{z}{2})^{l(k+1)}}{l!(k(l+p+1)-s)!}. \end{aligned} \tag{2.16}$$

Note that for $k = s = 1$ the relation (2.16) reduces to property 8) in (2.4). The right hand side of (2.16) can be considered as another generalization of the modified Bessel function since it is easy to reduce it to (1.2) when $k = s = 1$.

The function (2.16) again can be expressed in terms of the Wright function (2.6) as follows

$$\left(\frac{z}{2}\right)^{nk+k-s} \sum_{r=0}^{\infty} \frac{(\frac{z}{2})^{r(k+1)}}{r!\Gamma(kr+k(n+1)+1-s)}$$

$$= \left(\frac{z}{2}\right)^{nk+k-s} W\left(\frac{z^{k+1}}{2^{k+1}}, k, k(n+1)-s+1\right),$$

where n is positive.

3. Relations to other special functions

The paper by Kilbas, Saigo and Trujillo [5] deals with the generalized Wright function defined for $z \in C$, $a_j, b_j \in C$, $\alpha_j,\beta_j \in R\,(\alpha_j,\beta_j \neq 0,\ i = 1,..,p,\ j = 1,...,q)$ by the series

$$\psi_{p,q}(z) = \psi_{p,q}\left(\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_i,\beta_i)_{1,q} \end{array} \mid z \right) = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i r)}{\prod_{j=1}^{q}\Gamma(b_i+\beta_i r)}\frac{z^r}{r!} \tag{3.1}$$

which was introduced by Wright [17] for $p = 1, q = 2$. It is known that if

$$\Delta = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1, \tag{3.2}$$

then the series (3.1) is absolutely convergent for all $z \in C$.

In terms of the generalized Wright function (3.1) our generalized modified Bessel function (2.9) can be expressed as follows:

$$\begin{aligned} \tilde{I}_n^{k,s}(z) &= \left(\frac{z}{2}\right)^{n+k-s} \sum_{r=0}^{\infty} \frac{\left[\left(\frac{z}{2}\right)^{k+1}\right]^r}{r!} \frac{\Gamma(1+r)}{\Gamma(n+r+1)\Gamma(k-s+kr)} \\ &= \left(\frac{z}{2}\right)^{n+k-s} \psi_{1,2}\left(\begin{array}{c} (1,1) \\ (n+1,1)(k-s,k) \end{array} \middle| \frac{z^{k+1}}{2^{k+1}}\right), \end{aligned}$$

with $p = 1$, $a_i = 1$, $\alpha_i = 1$, $q = 2$, $b_1 = n+1$, $b_2 = 1$, $\beta_1 = k-s$, $\beta_2 = k$ in (3.1). In our case $\Delta = 1+k-1 > -1$ and by (3.2) the series (2.9) is absolutely convergent for all $z \in C$.

The generalized Wright function (3.1) is a special case of the so-called Fox‘s H-function (see [5], [6], [14]), which is widely used in Statistics and Queueing theory (see [10], [14]).

Note also that Kiryakova [7] introduced the so-called *multi-index Mittag-Leffler functions*

$$E_{(\beta_1,\ldots,\beta_m),(\mu_1,\ldots,\mu_m)}(z) \tag{3.3}$$

$$= \sum_{r=0}^{\infty} \varphi_r z^r = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\mu_1+\beta_1 r)...\Gamma(\mu_m+\beta_m r)}, \quad z \in C,\ m \geq 1,$$

where $z \in C$, $m \geq 1$ is an integer, $\beta_1, ..., \beta_m > 0$ and $\mu_1, ..., \mu_m$ are arbitrary real numbers.

For $m = 1$, the above function reduces to the classical Mittag-Leffler function

$$E_{\beta,\mu}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\mu+\beta r)}, \quad z \in C, \quad \mu > 0, \quad \beta > 0. \tag{3.4}$$

As shown in [7], the radius of convergence of series (3.3), by the Cauchy-Hadamard formula, is R, where

$$\frac{1}{R} = \lim_{r\to\infty} \sqrt[r]{|\varphi_r|} = \lim_{r\to\infty} \sup \left[\prod_{i=1}^{m} \Gamma(\mu_i+\beta_i r)\right]^{-\frac{1}{r}} = 0, \tag{3.5}$$

so it is an entire function. Indeed, by the Stirling's formula for large $r > 0$

$$\Gamma(r) \sim \sqrt{2\pi} r^{r-\frac{1}{2}} e^{-r},$$

$$\Gamma(\mu_i+\beta_i r) \sim \sqrt{2\pi}(\mu_i+\beta_i r)^{\mu_i+\frac{r}{\beta_i}-\frac{1}{2}} e^{-\mu_i} e^{\beta_i r}$$

and

$$\frac{1}{R} = \lim_{r\to\infty} \prod_{i=1}^{m} \left[(r\beta_i)^{-\beta_i} e^{\beta_i}\right] = \lim_{r\to\infty} r^{-(\beta_1+...+\beta_m)} e^{(\beta_1+...+\beta_m)} \beta_1^{-\beta_1} ... \beta_m^{-\beta_m} = 0.$$

It is clear that *our generalized modified Bessel function (2.9) can be expressed in terms of the multi-index Mittag-Leffler function* (3.3) as follows:

$$\tilde{I}_n^{k,s} = \left(\frac{z}{2}\right)^{n+k-s} \sum_{r=0}^{\infty} \left[\left(\frac{z}{2}\right)^{k+1}\right]^r \frac{1}{\Gamma(n+1+r)\Gamma(k-s+kr)}$$

$$= E_{(1,k),(n+1,k-s)}\left(\frac{z^{k+1}}{2^{k+1}}\right). \tag{3.6}$$

In Kiryakova's paper [7] various relationships between the multi-index Mittag-Leffler function (3.3) and other special functions, such as Fox's H-functions, Bessel-Maitland functions, Struve and Lommel functions, as well as the generalized fractional calculus [6], can be found.

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